REGULARITY CRITERIA FOR INTEGRAL AND MEROMORPHIC FUNCTIONS

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1. **Introduction.** In this paper we shall consider functions $f(z)$ which are meromorphic in the plane (hereinafter called meromorphic). In particular we consider integral functions. Throughout the paper we shall assume familiarity with the standard notation of the Nevanlinna theory,

$$T(r) = T(r,f),\ N(r,a),\ m(r,a),\ \delta(a,f)\ \cdots$$

and with the first fundamental theorem (see e.g. [7]). We define

$$M(r) = M(r,f) = \max|f(z)| \qquad (|z| = r),$$
$$\mu(r) = \mu(r,f) = \min|f(z)| \qquad (|z| = r),$$

using $\mu(r)$ instead of $m(r)$ for the minimum modulus to avoid confusion with the schmiegungsfunktion $m(r,f)$. We shall assume that $f(z)$ is transcendental i.e. that

$$\log r = o(T(r)) \qquad (r \to \infty)$$

and also that $f(0)=1$. It is easily seen in the sequel that this involves no loss of generality.

If $f(z)$ is an integral function then for r sufficiently large [7, p. 18]

$$T(r,f) \leqq \log M(r,f) \leqq \frac{R+r}{R-r} T(R,f) \qquad (0 < r < R). \tag{1.1}$$

From this it is easily deduced that the order or type of $f(z)$ is the same whether it is defined by $T(r,f)$ or $\log M(r,f)$. We note in particular that

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^\rho} > 0 \Leftrightarrow \liminf_{r\to\infty} \frac{\log M(r,f)}{r^\rho} > 0, \tag{1.2}$$

$$\liminf_{r\to\infty} \frac{T(r,f)}{r^\rho} < \infty \Leftrightarrow \liminf_{r\to\infty} \frac{\log M(r,f)}{r^\rho} < \infty. \tag{1.3}$$

If $f(z)$ is an integral function of order $\rho < 1$ then [13], [14]

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(1.4) $$\limsup_{r\to\infty} \frac{N(r,0)}{\log M(r)} \geqq \frac{\sin \pi\rho}{\pi\rho}.$$

Since $T(r,f) \geqq N(r,0) + O(1)$ we conclude

(1.5) $$\limsup_{r\to\infty} \frac{T(r)}{\log M(r)} \geqq \frac{\sin \pi\rho}{\pi\rho}.$$

Consider the integral function $F(z)$ with real negative zeros for which $(0 < \rho < 1)$

(1.6) $$n(r,0) \sim Ar^{\rho} \qquad (r \to \infty).$$

Then, as is well known (see e.g. [2])

$$N(r,0) \sim \frac{A}{\rho} r^{\rho} \qquad (r \to \infty),$$

$$\log M(r,F) = \log F(r) \sim \frac{A\pi}{\sin \pi\rho} r^{\rho} \qquad (r \to \infty),$$

$$\begin{aligned} T(r,F) &= m(r,F) \sim \frac{1}{2\pi} \frac{A\pi}{\sin \pi\rho} r^{\rho} \int_{-\pi}^{\pi} (\cos \rho\theta)^{+}\, d\theta, \\ &= \frac{A}{\rho} r^{\rho} \qquad (r \to \infty) \quad \left(0 < \rho \leqq \frac{1}{2}\right) \\ &= \frac{A}{\rho \sin \pi\rho} r^{\rho} \qquad (r \to \infty) \quad \left(\frac{1}{2} < \rho < 1\right). \end{aligned}$$

Thus (1.4) is best possible for $0 < \rho < 1$, and (1.5) is best possible for $0 < \rho \leqq \frac{1}{2}$.

2. The classical Wiman-Heins theory [8] and its extensions by Kjellberg [9], [10] lead one to expect that the integral functions which only just attain the growth demanded by (1.4) and (1.5) would have regular growth. We have the following theorems.

THEOREM 1. *Let $f(z)$ be an integral function with $f(0) = 1$ such that for some ρ, $0 < \rho \leqq \frac{1}{2}$,*

$$\pi\rho T(r) \leqq \sin \pi\rho \log M(r)$$

for all $r > 0$. Then

$$\beta = \liminf_{r\to\infty} \frac{T(r)}{r^{\rho}} > 0.$$

If, further, $\beta < \infty$, then

$$T(r) \sim \beta r^{\rho} \qquad (r \to \infty).$$

THEOREM 2. *Let $f(z)$ be an integral function with $f(0) = 1$ and such that for* *some* ρ, $0<\rho<1$

$$\pi\rho\, N(r,0) \leqq \sin\pi\rho \log M(r) \tag{2.1}$$

for all $r>0$. *Then*

$$\beta' = \liminf_{r\to\infty} \frac{\log M(r)}{r^{\rho}} > 0.$$

If, further, $\beta'<\infty$, *then*

$$\log M(r) \sim \beta' r^{\rho} \qquad (r\to\infty),$$

$$N(r,0) \sim \frac{\beta' \sin\pi\rho}{\pi\rho} \qquad (r\to\infty).$$

It is of interest to state the following corollary, which is implicit in some recent work of Edrei [3] (see also [6]).

COROLLARY 1. *Let* $f(z)$ *be an integral function of lower order* λ, $0<\lambda<1$, *then*

$$\limsup_{r\to\infty} \frac{T(r)}{\log M(r)} \geqq \limsup_{r\to\infty} \frac{N(r,0)}{\log M(r)} \geqq \frac{\sin\pi\lambda}{\pi\lambda}.$$

Proof. The first inequality is immediate. To prove the second let ρ be any number greater than λ. Then

$$\liminf_{r\to\infty} \frac{\log M(r)}{r^{\rho}} = 0.$$

Thus by Theorem 2 there exists a sequence $\{r_\nu\}$ say, of values of r tending to infinity, such that

$$\pi\rho N(r_\nu,0) > \sin\pi\rho \log M(r_\nu),$$

i.e.

$$\limsup_{r\to\infty} \frac{N(r,0)}{\log M(r)} \geqq \frac{\sin\pi\rho}{\pi\rho}.$$

The result follows on letting $\rho\to\lambda$.

If $\rho=1$ the condition (2.1) implies that $f(z)$ has 0 as a Picard (and a fortiori as a Borel) exceptional value. For such functions

$$\log M(r) \sim \alpha r^{n} \qquad (r\to\infty)$$

for some $\alpha>0$ and positive integer n. Thus

$$\beta = \liminf_{r\to\infty} \frac{\log M(r)}{r} > 0$$

and if $\beta<\infty$ then

$$\log M(r) \sim \beta r \qquad (r \to \infty).$$

Theorem 1 can thus be considered as an extension of this result to fractional orders.

Theorem 1 is easily deduced from Theorem 2 as follows: Suppose that for $r \geqq r_0$

$$\pi\rho T(r) \leqq \sin \pi\rho \log M(r).$$

Then, by the first fundamental theorem, since $f(0) = 1$

$$\pi\rho\, N(r,0) \leqq \pi\rho\, T(r) \leqq \sin \pi\rho \log M(r). \tag{2.2}$$

Thus by Theorem 2

$$\beta' = \liminf_{r\to\infty} \frac{\log M(r)}{r^\rho} > 0$$

and so by (1.2)

$$\beta = \liminf_{r\to\infty} \frac{T(r)}{r^\rho} > 0.$$

Also if $\beta < \infty$ then by (1.3) $\beta' < \infty$. Hence by Theorem 2

$$\sin \pi\rho \log M(r) \sim \beta' \sin \pi\rho\, r^\rho \qquad (r \to \infty),$$

$$\pi\rho\, N(r,0) \sim \beta' \sin \pi\rho\, r^\rho \qquad (r \to \infty)$$

and thus by the inequality (2.2)

$$T(r) \sim \frac{\beta' \sin \pi\rho}{\pi\rho} \qquad (r \to \infty)$$

as required.

3. For the case $\rho > \frac{1}{2}$ it is a conjecture of Paley [12] that for an integral function of order ρ

$$\limsup_{r\to\infty} \frac{T(r)}{\log M(r)} \geqq \frac{1}{\pi\rho}.$$

The example of §1 shows that the result would be sharp for $\frac{1}{2} < \rho < 1$. The Mittag-Leffler functions [7, p. 19] show that it would be sharp for $\rho > 1$. This conjecture is unproved, though Gol'dberg, [5], has shown that it is true with the additional assumption that there exists a θ for which

$$\log |f(re^{i\theta})| \sim \log M(r) \qquad (r \to \infty).$$

It is clear from our proof of Theorem 1 that it remains true for $\frac{1}{2} < \rho < 1$.

If the above conjecture of Paley is correct, however, the theorem would be vacuously true. Unfortunately the present results shed no light on the conjecture.

4. Theorem 1 can be extended, at any rate in part, to meromorphic functions. We have

THEOREM 3. *Let $f(z)$ be meromorphic with $f(0) = 1$ and such that for some ρ, $0 < \rho < \frac{1}{2}$,*

$$\pi\rho\, T(r) \leqq \sin\pi\rho \log M(r) + \pi\rho\cos\pi\rho\, N(r, \infty)$$

for all $r > 0$. Then

$$\beta = \liminf_{r\to\infty} \frac{T(r)}{r^\rho} > 0.$$

If, further, $\beta < \infty$ then

$$\alpha = \limsup_{r\to\infty} \frac{T(r)}{r^\rho} < \infty.$$

Theorem 3 is an immediate corollary of the following theorem.

THEOREM 4. *Let $f(z)$ be meromorphic in the plane and such that for some ρ, $0 < \rho < 1$, either*

$$\pi\rho\, N(r,0) \leqq \sin\pi\rho \log M(r) + \pi\rho\cos\pi\rho\, N(r, \infty) \tag{4.1}$$

or

$$\sin\pi\rho \log \mu(r) \leqq \pi\rho\cos\pi\rho\, N(r,0) - \pi\rho N(r, \infty) \tag{4.2}$$

for all $r > 0$. Then

$$\beta = \liminf_{r\to\infty} \frac{T(r)}{r^\rho} > 0.$$

If, further, $\beta < \infty$ then

$$\alpha = \limsup_{r\to\infty} \frac{T(r)}{r^\rho} < \infty.$$

REMARKS 1. Condition (4.2) is just condition (4.1) applied to $F(z) = (f(z))^{-1}$, and so it suffices just to consider (4.1).

2. The inequality (4.2) and its conclusion have been used by Ostrovskii [11] to show that for a meromorphic function of lower order $\lambda < \frac{1}{2}$,

$$\limsup_{r\to\infty} \frac{\log^+ \mu(r,f)}{T(r)} \geqq \pi\lambda(\operatorname{cosec}\pi\lambda)(\cos\pi\lambda - 1 + \delta(\infty)).$$

The result is sharp. An example to show this is easily constructed with the method of [7, p. 117].

3. It is an open question whether under the hypotheses of Theorems 3 and 4 we can conclude that $\alpha = \beta$ i.e. that $f(z)$ has perfectly regular growth in the sense of Valiron.

In the proof of Theorem 4 we can prove the following slightly more general theorem:

THEOREM 5. *Let $f(z)$ be meromorphic and such that, for some ρ, $0 < \rho < 1$, given $\varepsilon > 0$*

$$\text{(4.3)} \qquad \int_{r_1}^{r_2} (\pi\rho\, N(r,0) - \sin \pi\rho \log M(r) - \pi\rho \cos \pi\rho\, N(r, \infty)) \frac{dr}{r^{1+\rho}} < \varepsilon,$$

for all $r_2 > r_1 > r(\varepsilon)$ or

$$\text{(4.4)} \qquad \pi\rho\, N(r,0) \leqq \sin \pi\rho \log M(r) + \pi\rho \cos \pi\rho\, N(r, \infty) + O(\log r) \qquad (r \to \infty).$$

If $\beta < \infty$ then $\alpha < \infty$ and if (4.4) holds $\beta > 0$.

5. The proof of Theorem 4 uses results similar to those in [6]. We also use the techniques developed by Kjellberg.

LEMMA 1. *Let*

$$F(z) = \frac{F_1(z)}{F_2(z)} = \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) \Big/ \prod_{m=1}^{\infty} \left(1 - \frac{z}{b_m}\right), \qquad a_n > 0,\ b_m > 0,$$

be meromorphic and of order less than one. Then there exist constants K, k, depending only on $F(z)$ satisfying $0 < k < K < \infty$, such that for any $r_2 > r_1 > 0$, $0 < \rho < 1$,

$$\int_{r_1}^{r_2} (\pi\rho\, N(r,0) - \sin \pi\rho \log |F(r)| - \pi\rho \cos \pi\rho\, N(r, \infty)) \frac{dr}{r^{1+\rho}}$$

$$> k\, \frac{T(r_1, F)}{r_1^{\rho}} - K\, \frac{T(2r_2, F)}{r_2^{\rho}}.$$

Proof. Let C be the contour consisting of the line segments $r_1 \leqq t \leqq r_2$, $-r_2 \leqq t \leqq -r_1$ and the semicircles $|z| = r_1$, $0 < \arg z < \pi$, and $|z| = r_2$, $0 < \arg z < \pi$, with indentations, of radius δ say, around the zeros and poles of $F(z)$. Consider

$$\int_C \frac{\log F(z)}{z^{1+\rho}}\, dz.$$

We consider that branch of $z^{1+\rho}$ which is real for $z > 0$ and that branch of $\log F(z)$ for which $\log F_1(z)$ is real for $z > 0$ and $\log F_2(z)$ real for $z < 0$. Since $F(z)$ is meromorphic $\log F(z)$ has only logarithmic singularities at the zeros and poles of $F(z)$. The contribution to the integrand along any indentation is therefore $O(\delta\log 1/\delta)$ as $\delta\to 0$. Since $F(z)$ is analytic inside the contour we obtain on letting $\delta\to 0$,

$$0=\int_{r_1}^{r_2}(\log F(r)-e^{-i\pi\rho}\log F(-r))\frac{dr}{r^{1+\rho}}$$

$$+\frac{i}{r_2^{\rho}}\int_0^{\pi}e^{-i\rho\theta}\log F(r_2e^{i\theta})\,d\theta-\frac{i}{r_1^{\rho}}\int_0^{\pi}e^{-i\rho\theta}\log F(r_1e^{i\theta})\,d\theta.$$

On multiplying through by $e^{i\pi\rho}$ and taking real parts we obtain,

$$\int_{r_1}^{r_2}(\pi\, n(r,0)-\sin\pi\rho\log|F(r)|-\pi\cos\pi\rho\, n(r,\infty))\frac{dr}{r^{1+\rho}}$$

$$=r_1^{-\rho}P(r_1)-r_2^{-\rho}P(r_2)$$

where

$$P(r)=-\int_0^{\pi}(\cos\rho(\pi-\theta)\log|F(re^{i\theta})|-\sin\rho(\pi-\theta)\arg F(re^{i\theta}))\,d\theta.$$

Now

$$\int_{r_1}^{r_2}\frac{n(r,0)dr}{r^{1+\rho}}=r_2^{-\rho}N(r_2,0)-r_1^{-\rho}N(r_1,0)+\rho\int_{r_1}^{r_2}\frac{N(r,0)dr}{r^{1+\rho}}.$$

Thus we obtain

(5.1)
$$\int_{r_1}^{r_2}(\pi\rho N(r,0)-\sin\pi\rho\log|F(r)|-\pi\rho\cos\pi\rho\, N(r,\infty))\frac{dr}{r^{1+\rho}}$$

$$=r_1^{-\rho}Q(r_1)-r_2^{-\rho}Q(r_2)$$

where

$$Q(r)=\pi N(r,0)-\pi\cos\pi\rho\, N(r,\infty)+P(r).$$

An application of Jensen's theorem [7, formula (1.5)] yields

$$Q(r)=\pi(1-\cos\pi\rho)N(r,\infty)$$

$$+\int_0^{\pi}(1-\cos\rho(\pi-\theta))\log|F(re^{i\theta})|$$

$$+\sin\rho(\pi-\theta)\arg F(re^{i\theta})\,d\theta,$$

since $F(z)$ is symmetric with respect to the real axis. Now

$$\pi(1-\cos\pi\rho)N(r,\infty)<2\pi T(r),$$

$$\int_0^\pi (1-\cos\rho(\pi-\theta)\log|F(re^{i\theta})| < 2\pi T(r),$$

$$\int_0^\pi \sin\rho(\pi-\theta)\arg F(re^{i\theta})\,d\theta < [\pi n(r,0)+\pi n(r,\infty)]\int_0^\pi \sin\rho(\pi-\theta)\,d\theta$$
$$< \pi^2[n(r,0)+n(r,\infty)].$$

Now

$$n(r,0)\log 2 = n(r,0)\int_r^{2r}\frac{dt}{t} < \int_r^{2r}\frac{n(t,0)dt}{t} < N(2r,0) < T(2r). \tag{5.2}$$

Similarly for $n(r,\infty)$, and so we obtain

$$Q(r) < 4\pi T(r) + \frac{2\pi^2}{\log 2}T(2r) < KT(2r). \tag{5.3}$$

The left-hand inequality is not so immediate but it follows from the fact that $\psi(\theta)=1-\cos\rho(\pi-\theta)$ is a decreasing function of θ for $0<\theta<\pi$ and that

$$m(r,F) = \frac{1}{\pi}\int_0^\gamma \log|F(re^{i\theta})|\,d\theta,$$

$$m\left(r,\frac{1}{F}\right) = \frac{1}{\pi}\int_\gamma^\pi \log|F(re^{i\theta})|\,d\theta$$

for some $\gamma=\gamma(r)$ satisfying $0\leqq\gamma\leqq\pi$.

We prove the inequality only in the case $0\leqq\gamma\leqq\pi/3$, the cases when $\pi/3<\gamma\leqq 2\pi/3$ and $2\pi/3<\gamma\leqq\pi$ being similar. Since

$$\sin\rho(\pi-\theta)\arg F(re^{i\theta}) \geqq 0$$

for $0\leqq\theta\leqq\pi$ we have

$$Q(r) \geqq \pi(1-\cos\pi\rho)N(r,\infty) + \int_0^\pi (1-\cos\rho(\pi-\theta))\log|F(re^{i\theta})|\,d\theta$$
$$\geqq \pi(1-\cos\pi\rho)N(r,\infty) + \pi(1-\cos\rho(\pi-\gamma))m(r,\infty)$$
$$+\int_\gamma^\pi (1-\cos\rho(\pi-\theta))\log|F(re^{i\theta})|\,d\theta.$$

Now $\log|F(re^{i\theta})|$ is a decreasing function of θ and is less than zero for $\gamma<\theta\leqq\pi$. Thus

$$\int_{2\pi/3}^\pi \log|F(re^{i\theta})|\,d\theta < \frac{1}{3}\int_\gamma^\pi \log|F(re^{i\theta})|\,d\theta = -\frac{\pi}{3}m(r,0). \tag{5.4}$$

Hence,

$$
\begin{aligned}
Q(r) \geqq\ & \pi(1-\cos\rho(\pi-\gamma))T(r,F) \\
& + \left\{\int_{\gamma}^{2\pi/3} + \int_{2\pi/3}^{\pi}\right\}(1-\cos\rho(\pi-\theta))\log\left|(F\,re^{i\theta})\right|d\theta \\
\geqq\ & \pi(1-\cos\rho(\pi-\gamma))T(r,F) + (1-\cos\rho(\pi-\gamma))\int_{\gamma}^{2\pi/3}\log\left|F(re^{i\theta})\right|d\theta \\
& + \left(1-\cos\frac{\pi\rho}{3}\right)\int_{2\pi/3}^{\pi}\log\left|F(re^{i\theta})\right|d\theta \\
=\ & \pi(1-\cos\rho(\pi-\gamma))T(r,F) - \pi(1-\cos\rho(\pi-\gamma))m(r,0) \\
& + \left(\text{cosp}\,\rho(\pi-\gamma)-\cos\frac{\pi\rho}{3}\right)\int_{2\pi/3}^{\pi}\log\left|F(re^{i\theta})\right|d\theta \\
>\ & \pi(1-\cos\rho(\pi-\gamma))T(r,F) - \pi(1-\cos\rho(\pi-\gamma))m(r,0) \\
& + \frac{\pi}{3}\left(\cos\frac{\pi\rho}{3} - \cos\rho(\pi-\gamma)\right)m(r,0)
\end{aligned}
$$

by (5.4). Therefore by the first fundamental theorem

$$
\begin{aligned}
Q(r) &\geqq \frac{\pi}{3}\left[\cos\frac{\pi\rho}{3} - \cos\rho(\pi-\gamma)\right]T(r,F) \\
&\geqq \frac{\pi}{3}\left(\cos\frac{\pi\rho}{3} - \cos\frac{2\pi\rho}{3}\right)T(r,F)
\end{aligned}
$$

since $0 \leqq \gamma < \pi/3$. This completes the proof of the lemma.

6. **Proof of Theorem 4.** If $f(z)$ has only finitely many zeros and poles then, since we are assuming that $f(z)$ is transcendental

$$f(z) = \frac{P_1(z)}{P_2(z)}\exp\phi(z),$$

where P_1, P_2 are polynomials and $\phi(z)$ is an integral function. From this we conclude that $f(z)$ has lower order at least 1 and so

$$\beta = \liminf_{r\to\infty}\frac{T(r)}{r^{\rho}} = \infty$$

for any ρ, $0 < \rho < 1$, and so the theorem is proved.

Now, following Kjellberg we choose R sufficiently large so that $f(z)$ has N zeros and M poles in $|z| < R$ where $\max(M,N) > 0$, R being suitably chosen later. We denote zeros by a_n and poles by b_m. Let

$$f_1(z) = \prod_{n=1}^{N}\left(1-\frac{z}{a_n}\right)\Big/\prod_{m=1}^{M}\left(1-\frac{z}{b_m}\right),$$

$$f_2(z) = \prod_{n=1}^{N}\left(1 + \frac{z}{|a_n|}\right) \Big/ \prod_{m=1}^{M}\left(1 - \frac{z}{|b_m|}\right),$$

and define $f_3(z)$ by

$$f(z) = f_1(z) f_3(z). \tag{6.1}$$

Then for $0 < r < \frac{1}{2}R$, [3, Lemma A]

$$\log |f_3(re^{i\theta})| < \frac{14T(2R)}{R} r. \tag{6.2}$$

Now

$$T(R,f_2) \leqq N(R,0) + N(R,\infty) + \sum_{n=1}^{N} \log\left(1 + \frac{R}{|a_n|}\right) + \sum_{m=1}^{M} \log\left(1 + \frac{R}{|b_m|}\right) \tag{6.3}$$

and

$$\begin{aligned}\sum_{n=1}^{N} \log\left(1 + \frac{R}{a_n}\right) &= \int_0^R \log\left(1 + \frac{r}{t}\right) dn(t,0) \\ &< n(R,0)\log 2 + \int_0^R \frac{R}{R+t} \frac{n(t,0)dt}{t} \\ &< T(2R,f) + N(R,0) \qquad \text{by (5.2)} \\ &< 2T(2R,f).\end{aligned}$$

Thus from (6.3) we obtain

$$T(R,f_2) \leqq 6T(2R,f). \tag{6.4}$$

Also, by a result of Edrei [3, formula 8.4], we have for $r \leqq \frac{1}{2}R$

$$T(r,f) \leqq T(r,f_2) + \frac{14r}{R} T(2R,f). \tag{6.5}$$

We now apply Lemma 1 to $f_2(z)$, which satisfies the hypotheses, to obtain, for any r_1, r_2, $0 < r_1 < r_2 < R$,

$$\begin{aligned}\int_{r_1}^{r_2} (\pi\rho N(r,0) - \sin\pi\rho \log|f_2(r)| - \pi\rho\cos\pi\rho N(r,\infty)) \frac{dr}{r^{1+\rho}} \\ > k\frac{T(r_1,f_2)}{r_1^{\rho}} - K\frac{T(2r_2,f_2)}{r_2^{\rho}},\end{aligned}$$

where k, K depend only on f_2, i.e. on f. Thus by our hypothesis (4.1)

$$\begin{aligned}0 \geqq \sin\pi\rho \int_{r_1}^{r_2} (\log|f_2(r)| - \log M(r,f)) \frac{dr}{r^{1+\rho}} \\ + k\frac{T(r_1,f_2)}{r_1^{\rho}} - K\frac{T(2r_2,f_2)}{r_2^{\rho}}.\end{aligned} \tag{6.6}$$

But from (6.1),

$$\log M(r,f) \leqq \log M(r,f_1) + \log M(r,f_3) \leqq \log|f_2(r)| + \log M(r,f_3)$$

i.e.

$$\log|f_2(r)| - \log M(r,f) \geqq -\log M(r,f_3).$$

Thus for $r_2 \leqq \frac{1}{2}R$ we obtain by (6.2),

$$\sin \pi\rho \int_{r_1}^{r_2} (\log|f_2(r)| - \log M(r,f))\frac{dr}{r^{1+\rho}} > -14 \sin\pi\rho \, \frac{T(2R)}{R} \int_{r_1}^{r_2} \frac{dr}{r^{\rho}}$$

$$= \frac{-14 \sin\pi\rho}{1-\rho} \frac{T(2R)}{R}\left\{r_2^{1-\rho} - r_1^{1-\rho}\right\}$$

$$\frac{-14\sin\pi\rho}{1-\rho} \; \frac{T(2R)}{R} r_2^{1-\rho}.$$

If we now choose $r_2 = \frac{1}{2}R$ we obtain from (6.6)

$$0 \geqq k\frac{T(r_1,f_2)}{r_1^{\rho}} - 2^{\rho}K\frac{T(R,f_2)}{R^{\rho}} - 2^{\rho-1}\cdot 14 \,\frac{\sin\pi\rho}{1-\rho}\,\frac{T(2R,f)}{R^{\rho}}.$$

We now use the estimates (6.4) and (6.5) to obtain

$$0 \geqq k\frac{T(r_1,f)}{r_1^{\rho}} - 14\left(\frac{r_1}{R}\right)^{1-\rho} \frac{T(2R,f)}{R^{\rho}} - \left(6.2^{\rho}K + \frac{14.2^{\rho-1}\sin\pi\rho}{1-\rho}\right)\frac{T(2R,f)}{R^{\rho}}.$$

Finally since $r_1 < \frac{1}{2}R$ we have, for some suitable constant $K_1 > 0$.

(6.7) $$0 \geqq k\frac{T(r_1,f)}{r_1^{\rho}} - K_1\,\frac{T(2R,f)}{(2R)}.$$

This holds for all R and all $r_1 < \frac{1}{2}R$.

7. Theorem 4 now follows from (6.7). Suppose that r_1 is fixed, then by the definition of β there exist arbitrarily large values of R such that for any $\varepsilon > 0$

$$T(2R,f) < (\beta+\varepsilon)(2R)^{\rho}.$$

Thus for any $\varepsilon > 0$

(7.1) $$k\,\frac{T(r_1,f)}{r_1^{\rho}} \leqq K_1(\beta+\varepsilon).$$

The left-hand side of this inequality is a fixed positive number and so if $\beta = 0$ this gives a contradiction. Thus

$$\beta = \liminf_{r\to\infty} \frac{T(r)}{r^{\rho}} > 0.$$

Also (7.1) holds for any r_1. Thus if $\beta < \infty$ we obtain

$$\alpha = \limsup_{r\to\infty} \frac{T(r)}{r^\rho} \leqq \frac{K_1\beta}{k} < \infty.$$

This completes the proof of Theorem 4.

We note that for a given $f(z)$ explicit values of K_1 and k could be calculated. This, however, sheds no light on the more interesting question of whether or not $\alpha = \beta$ under the hypotheses of Theorem 4.

If instead of (4.1) we use the assertions (4.3) and (4.4) we obtain instead of (6.7) the assertions

$$\varepsilon \geqq k\frac{T(r_1,f)}{r_1^\rho} - K_1\,\frac{T(2R,f)}{(2R)^\rho}, \qquad r_1 > r(\varepsilon)$$

and

$$\frac{K_2 \log r_1}{r_1^\rho} \geqq k\frac{T(r_1,f)}{r_1^\rho} - K_1\,\frac{T(2R,f)}{(2R)^\rho}$$

respectively for some $K_2 < \infty$. The conclusions then follow as before since we are assuming that $f(z)$ is transcendental. Thus Theorem 5 is proved.

8. It remains to prove Theorem 2. By Theorem 4 and (1.1) we have $\beta' > 0$, and if $\beta' < \infty$, then

$$\alpha' = \limsup_{r\to\infty} \frac{\log M(r)}{r^\rho} < \infty \tag{8.1}$$

and we have to show that $\alpha = \beta$. Now $f(z)$ has genus zero by (8.1). Let

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right),$$

$$F(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{|a_n|}\right).$$

Then, [8], p. 204, we have

$$\log M(r,f) \leqq \log F(r), \tag{8.2}$$

Hence by (2.1)

$$N(r,0) \leqq \frac{\sin \pi\rho}{\pi\rho}\log F(r). \tag{8.3}$$

Also from (8.1)

$$\alpha_1 = \limsup_{r\to\infty} \frac{\log F(r)}{r^\rho} < \infty.$$

Now $F(z)$ satisfies the hypotheses of Lemma 1, and so, with the notation of that lemma

$$\text{(8.4)} \qquad \int_{r_1}^{r_2} (\pi\rho N(r,0) - \sin\pi\rho \log F(r)) \frac{dr}{r^{1+\rho}} = \frac{Q(r_1)}{r_1^\rho} - \frac{Q(r_2)}{r_2^\rho}.$$

Now by (5.2),

$$Q(r) < KT(2r) < K \log F(2r)$$

and so from (8.4)

$$\alpha'' = \limsup_{r\to\infty} \frac{Q(r)}{r^\rho} < \infty.$$

By (8.3) we obtain

$$\frac{Q(r_1)}{r_1^\rho} \leqq \frac{Q(r_2)}{r_2^\rho}$$

for all $r_2 > r_1 > 0$. Now let $r_2 \to \infty$ through a suitable sequence of values so that $r_2^{-\rho}Q(r_2)$ tends to its lower limit β'' say. We obtain $\alpha'' \leqq \beta''$ and so $Q(r) \sim \alpha'' r^\rho (r \to \infty)$.

Applying this to (8.5) we see that given any $\varepsilon > 0$ there exists an $r(\varepsilon)$ such that for $r_2 > r_1 > r(\varepsilon)$,

$$\varepsilon > \int_{r_1}^{r_2} (\pi\rho N(r,0) - \sin\pi\rho \log F(r)) \frac{dr}{r^{1+\rho}} > -\varepsilon.$$

By our hypothesis (2.1) and (8.2)

$$-\varepsilon < \int_{r_1}^{r_2} (\pi\rho N(r,0) - \sin\pi\rho \log M(r,f)) \frac{dr}{r^{1+\rho}} \leqq 0$$

and so we obtain by subtraction that

$$0 \geqq \sin\pi\rho \int_{r_1}^{r_2} (\log F(r) - \log M(r,f)) \frac{dr}{r^{1+\rho}} \geqq -2\varepsilon.$$

Thus the integral

$$\int_0^\infty (\log F(r) - \log M(r,f)) \frac{dr}{r^{1+\rho}}$$

exists and is finite. Now,

$$\log F(r) = r \int_0^\infty \frac{n(t,0)dt}{t(t+r)} = r \int_0^\infty \frac{N(t,0)dt}{(t+r)^2}$$

and so we may write (8.3) as

$$r^{-\rho}N(r,0) \leqq \frac{\sin\pi\rho}{\pi\rho} r^{1-\rho} \int_0^\infty \frac{N(t,0)}{t^\rho} \frac{t^\rho dt}{(t+r)^2}$$

$$= \frac{\sin\pi\rho}{\pi\rho} \int_0^\infty \frac{N(t,0)}{t^\rho} \frac{(t/r)^{1+\rho}}{((t/r)+1)^2} \frac{dt}{t}.$$

We let $r = e^s$, $t = e^u$ and let $\psi(s) = r^{-\rho}N(r,0)$. Thus (8.3) becomes

$$\psi(s) \leqq \frac{\sin \pi\rho}{\pi\rho} \int_{-\infty}^{\infty} \psi(u) \frac{(e^{u-s})^{1+\rho}}{(e^{u-s}+1)^2}$$

i.e.

$$\psi(s) \leqq \int_{-\infty}^{\infty} \psi(u) K(u-s)\, du, \tag{8.5}$$

where

$$K(x) = \frac{\sin \pi\rho}{\pi\rho} e^{x(1+\rho)}(e^x + 1)^{-2}.$$

Convolution inequalities like (8.5) have been studied by Essén [4]. He has the following lemma.

Lemma 2. *Let $\psi(s)$ be bounded and slowly decreasing, i.e.*

$$\liminf_{x\to\infty} \liminf_{y-x\to 0;\, y>x} |\psi(y) - \psi(x)| \geqq 0.$$

If $K(x) \in \mathscr{L}(-\infty, \infty)$ and satisfies

$$\int_{-\infty}^{\infty} K(x)\, dx = 1,$$

$$\int_{-\infty}^{\infty} |x| K(x)\, dx < \infty,$$

$$\int_{-\infty}^{\infty} xK(x)\, dx = m \neq 0,$$

then the inequality (8.5) *implies that* $\lim_{x\to\infty} \psi(x)$ *exists.*

If we apply the lemma to $\psi(s) = r^{-\rho}N(r,0)$ as above, we obtain

$$N(r,0) \sim lr^{\rho} \qquad (r \to \infty)$$

for some l. We must show that ψ and K satisfy the hypothesis of the lemma. It is easy to verify that

$$\int_{-\infty}^{\infty} \frac{e^{x(1+\rho)}dx}{(e^x+1)^2} = \pi\rho(\operatorname{cosec} \pi\rho).$$

Differentiating with respect to ρ we obtain $(0 < \rho < 1)$

$$\int_{-\infty}^{\infty} \frac{xe^{x(1+\rho)}dx}{(e^x+1)^2} = \pi(\sin \pi\rho - \pi\rho \cos \pi\rho) \operatorname{cosec}^2 \pi\rho \neq 0.$$

The other condition on K is also clearly satisfied. It follows from (8.1) that $\psi(s)$ is bounded. Also

$$\psi(y) - \psi(x) = \frac{N(r_2)}{r_2^\rho} - \frac{N(r_1)}{r_1^\rho} > \frac{N(r_1)}{r_1^\rho}\left[\left(\frac{r_1}{r_2}\right)^\rho - 1\right].$$

As $r_2/r_1 \to 1$ and $r \to \infty$ the above expression tends to zero, and so ψ is slowly decreasing. Thus

$$N(r,0) \sim lr^\rho \qquad (r \to \infty)$$

and elementary Tauberian and Abelian arguments enable us to conclude that

$$n(r,0) \sim l\rho r^\rho \qquad (r \to \infty),$$

$$\log F(r) \sim \pi l\rho(\operatorname{cosec}\pi\rho)r^\rho \qquad (r \to \infty).$$

But from (8.4) we conclude by a well-known argument [1, §4] that

$$\lim r^{-\rho}(\log F(r) - \log M(r,f)) = 0$$

as $r \to \infty$ outside an open set, E say, of finite logarithmic length. Thus for $r \notin E$

$$\log M(r,f) \sim \pi l\rho(\operatorname{cosec}\pi\rho)r^\rho \qquad (r \to \infty).$$

But if $r \in E$ there exists $r_1 r_2 \notin E$ with $r_1 < r < r_2$ and such that $\log(r_2/r_1) \to 0$ $(r_1 \to \infty)$. Now $\log M(r,f)$ is a monotonic increasing function of r. Thus given $\varepsilon > 0$ we have, for r sufficiently large,

$$\log M(r,f) > \log M(r_1,f) \sim \pi l\rho(\operatorname{cosec}\pi\rho)r_1^\rho > \pi l\rho(\operatorname{cosec}\pi\rho)(1-\varepsilon)r^\rho$$

and

$$\log M(r,f) < \log M(r_2,f) \sim \pi l\rho(\operatorname{cosec}\pi\rho)r_2^\rho < \pi l\rho(\operatorname{cosec}\pi\rho)(1+\varepsilon)r^\rho.$$

Hence

$$\log M(r,f) \sim \pi l\rho(\operatorname{cosec}\pi\rho)r^\rho \qquad (r \to \infty).$$

Thus, by the definition of β',

$$\log M(r,f) \sim \beta' r^\rho \qquad (r \to \infty),$$

$$N(r,0) \sim \frac{\beta' \sin \pi\rho}{\pi\rho} r^\rho \qquad (r \to \infty),$$

which is the required result.

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